Using nonlinear saturated feedback to control chaos: The Hénon map

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We used nonlinear saturated feedback to regulate the motion of chaotic dynamical systems. We studied the case of the Hénon map by controlling both unstable equilibrium points and periodic orbits. A key issue addressed is that of the geometry of the saturated feedback.

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INTRODUCTION

In recent years, there has been a growing interest in the control of chaotic (both continuous- and discrete-time) systems [1–9]. Although chaos is a beneficial feature in heat- and mass-transport phenomena, in many situations it is an undesirable phenomenon which may lead to vibrations, irregular operation, and fatigue failure in mechanical systems [6]. Given a chaotic system, one would like to regulate its motion around a reference trajectory, commonly a low-period periodic orbit, in order to obtain improved performance. A central observation is that a chaotic attractor typically has embedded densely within it an infinite number of unstable periodic orbits, in addition to unstable equilibrium points [2].

Departing from existing theories on robust control of dynamical systems [10], it is possible, in principle, to construct a feedback control which steers the dynamic of a system to a desired target (equilibrium points, periodic orbits, etc.). However, in practice such feedback leads commonly to large, costly, and physically unfeasible alterations of the system [1]. Therefore most approaches to control chaos must assume that only small temporal perturbation of accessible parameters are allowed. From a control-theory viewpoint, the above implies a problem of control with bounded inputs [12].

Ott, Grebogi, and Yorke (OGY) [2] reported the first strategy to control chaos. The OGY methodology consists in applying feedback control actions only when the trajectory of the system is in a small neighborhood of a given objective trajectory, so that parameter perturbation boundedness is satisfied. Due to its local nature, OGY control leads to poor performance with large transient periods. Recently, Romeiras *et al.* [9] extended the OGY methodology, allowing for a more general choice of the feedback matrix. Such a feedback matrix is constructed in base-to-pole placement techniques to assure local asymptotic stability. In order to have small parameter perturbations, they used a discontinuous saturation function. As in the OGY methodology, the feedback is linear and local. We believe that local linear designs do not take advantage of all the stabilization capacities of nonlinear systems.

As a step to approach the control of a chaotic system with nonlinear feedback action, we study the control of the Hénon map [11]. As objective trajectories, we take both unstable fixed points and periodic orbits. Our feedback control is constructed in two stages. First, a nonlinear feedback is constructed which transforms the Hénon map to a linear system, its global attractor being the objective trajectory. Then, to satisfy the small parameter perturbation constraint, a saturation function is introduced.

UNBOUNDED CONTROL

Consider the Hénon map $x_{n+1}=F(x_n,u)$ [11]:

$$\begin{aligned} x_{1,n+1}&=x_{2,n} \ , \\ x_{2,n+1}&=-x_{2,n}^2+Bx_{1,n}+u \ , \end{aligned} \tag{1}$$

where u is an accessible parameter. It is well known that for certain values of the parameter set (B,u), the system (1) exhibits very complex behavior [1,4]. The feedback $u=u_n(x;v)=x_{2,n}^2-Bx_{1,n}+v$, where v is a *dummy* input, transforms the system (1) to the controllable input-output linear map (see [10], and references therein) $x_{1,n+2}=v$, with input v and output x_1. Given an arbitrary reference trajectory $\{r_n\}$, the controllability of $x_{1,n+2}=v$ implies that one can design a linear feedback $v=v_n(r_n)$ such that the closed-loop system $x_{1,n+2}=v_n(r_n)$ has $\{r_n\}$ as its global attractor. Therefore the system (1) with feedback $u=u_n(x;v)=x_{2,n}^2-Bx_{1,n}+v_n(r_n)$ has $\{r_n\}$ as its global attractor. By defining the error of tracking as $\varepsilon_n=x_{1,n}-r_n$, it is possible to see that $v=v_n(r_n)$ must be given by $v_n(r_n)=-g_1(x_{1,n}-r_n)-g_2(x_{2,n}-r_{n+1})+r_{n+2}$, where g_1 and g_2 are tunable parameters such that the matrix

$$\begin{bmatrix} 0 & 1 \\ -g_1 & -g_2 \end{bmatrix}$$

has its eigenvalues in the open unit circle.

BOUNDED CONTROL

Suppose that the parameter u is restricted to take only small controlling temporal perturbations δ^*. If we apply the feedback $u_n(x;v)$ to system (1), the input u may take arbitrary values, so that the small temporal perturbation constraint is not necessarily satisfied. Let $u=\bar{u}$ be a nominal input to system (1). Let us introduce the following saturation function $\mathcal{S}:\mathbb{R}\rightarrow[\bar{u}-\delta^*,\bar{u}+\delta^*]$:

$$\mathcal{S}(u)=\begin{cases}\bar{u}-\delta^* & \text{if } \bar{u}-\delta^*>u\\ u & \text{if } \bar{u}-\delta^*\le u\le\bar{u}+\delta^*\\ \bar{u}+\delta^* & \text{if } u>\bar{u}+\delta^* .\end{cases}$$

Then, $|\mathcal{S}(u_n(x;v))-\bar{u}|\le\delta^*$. In this way, we obtain a controlled map $x_{n+1}=F_{\text{sat}}(x_n)=F(x_n,\mathcal{S}(u_n(x;v)))$, which is continuous and piecewise smooth. At each time n, $\mathcal{S}$ induces a partition of $\mathbb{R}^2$ into three regions:

$$U_n^0=\{x\in\mathbb{R}^2 : \bar{u}-\delta^*\le u_n(x;v)\le\bar{u}+\delta^*\}$$

and

$$U_n^+(U_n^-)=\{x\in\mathbb{R}^2 : u_n(x;v)>\bar{u}+\delta^* \ (<\bar{u}-\delta^*)\} ,$$

with $U_n^0\cup U_n^+\cup U_n^-=\mathbb{R}^2$. The curves

$$S_n^+ \ (S_n^-)=\{x\in\mathbb{R}^2 : u_n(x;v)=\bar{u}+\delta^* \ (\bar{u}-\delta^*)\}$$

define the saturation boundaries. In U_n^0, $F_{\text{sat}}(x_n)$ behaves as the controlled linear system $x_{n+1}=v_n(x_n)$, while in U_n^+ (U_n^-), $F_{\text{sat}}(x_n)$ behaves as the noncontrolled system (1) with input $u=\bar{u}+\delta^*$ $(\bar{u}-\delta^*)$.

Let $\{r_n\}$ be a stationary set of the system (1) with input $u=\bar{u}$. It is not difficult to see that $r_n\in U_n^0$. In this way, the set $\{r_n\}$ is a (local) attractor of the map $x_{n+1}=F_{\text{sat}}(x_n)$. Thus $\{r_n\}$ can be attained by means of the saturated feedback $\mathcal{S}(u_n(x;v))$.

Consider the case of $B=0.3$ and $u=\bar{u}=1.29$. With this set of parameters, the system (1) has an unstable equilibrium point $x^*=\{0.8384,0.8384\}$ embedded in a chaotic attractor $\mathcal{A}$. Let $\{r_n\}=\{0.8384\}$ be the reference trajectory. Assume that $g_1=g_2=0$, which implies that the point x^*, as an equilibrium point of $x_{n+1}=F_{\text{sat}}(x_n)$, is stable with eigenvalues $\lambda_1=\lambda_2=0$. Thus trajectories with initial condition close enough to x^* have a transient period $\langle\tau\rangle$ of at most three. Figure 1 shows the chaotic attractor $\mathcal{A}$ and the embedded equilibrium point x^*. Also shown are the saturation curves S_n^+ and S_n^- (which are the same for each time n) for $\delta^*=0.1$. Observe that the set $\mathcal{A}\cap U_n^0$ is *destroyed* by the feedback $\mathcal{S}(u_n(x;v))$. In this way, iterations arriving close to x^* are immediately (in one or two iterations) stabilized by the controller. Following a procedure analogous to the one proposed by Ott, Grebogi, and Yorke [2], we calculated the transient period $\langle\tau\rangle$ as a function of the input bandwidth δ^*. Like in the results of Ott, Grebogi, and Yorke, the $\langle\tau\rangle-\delta^*$ behavior follows a power-law trend for small δ^*.

From numerical simulations, it is apparent that x^* is the unique attractor for $\delta^*\notin(0.092,0.185)$: trajectories starting in the set $\mathcal{A}\cap[U^+\cup U^-]$ converge asymptotically to x^*. However, at $\delta^*\simeq0.092$ a multistability appears: x^* coexists with a chaotic attractor $\mathcal{A}_1$ (which is not equal to $\mathcal{A}$). Figure 2 shows the geometry of the $x^*-\mathcal{A}_1$ coexistence: observe that $\mathcal{A}_1$ resembles $\mathcal{A}\cap[U^*\cup U^-]$. However, $\mathcal{A}_1$ and $\mathcal{A}\cap[U^+\cup U^-]$ cannot be the same set. In fact, if $\mathcal{A}_1=\mathcal{A}\cap[U^*\cup U^-]$,

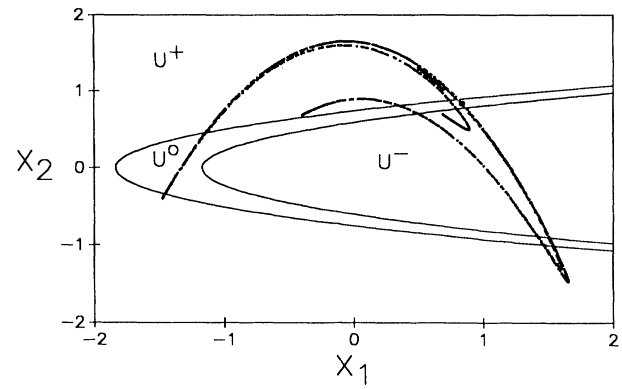

FIG. 1. Chaotic trajectory of the Hénon map for $\bar{u}=1.29$ and $B=0.3$. Also shown is the geometry of the saturated feedback $\mathcal{S}(u_n(x;v))$ for $\delta^*=0.1$. ● denotes the embedded equilibrium point $x^*=(0.8384,0.8384)$.

the embedding of x^* in $\mathcal{A}$ would imply that there are trajectories in $\mathcal{A}_1$ which converge asymptotically to x^*, so that $\mathcal{A}_1$ would not be an attractor. For $\delta^*>0.092$, $\mathcal{A}_1$ evolves continuously to disappear at $\delta^*\simeq0.185$ (crisis of the chaotic attractor) where it is absorbed by the nonsaturation region U_n^0.

Next, we consider the issue of robustness against parameter uncertainties. Assume that B_m is an estimate of B. If $B\neq B_m$, then the feedback $\mathcal{S}(u_n(x;v))$ does not make the system (1) equivalent to a linear system. Thus x^* is no longer an equilibrium point of the controlled system $x_{n+1}=F_{\text{sat}}(x_n)$. However, numerical simulation shows that the feedback $\mathcal{S}(u_n(x;v))$ is robust in the sense that the trajectories are steered close to x^*.

The system (1) has an unstable two-period periodic orbit $P_2\simeq[(1.3104,-0.6104),(-0.6104,1.3104)]$, embedded in $\mathcal{A}$. Let $\{r_n\}=\{r_1,r_2\}=\{1.3104,-0.6104\}$ be the reference trajectory. Since $\{r_n\}$ is not a constant number, S_n^+ and S_n^- change at each time n. Figure 3 shows the evolution of the iterates for $\delta^*=0.025$. Like in the equilibrium-point case, the trajectory converges to the periodic orbit P_2 after a transient period induced by the iterates in $U_n^+\cup U_n^-$.

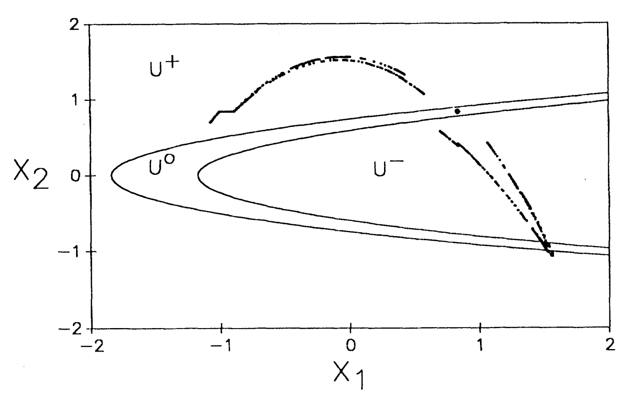

FIG. 2. Geometry of the coexistence of the (controlled) attractor x^* and the chaotic attractor $\mathcal{A}_1$.

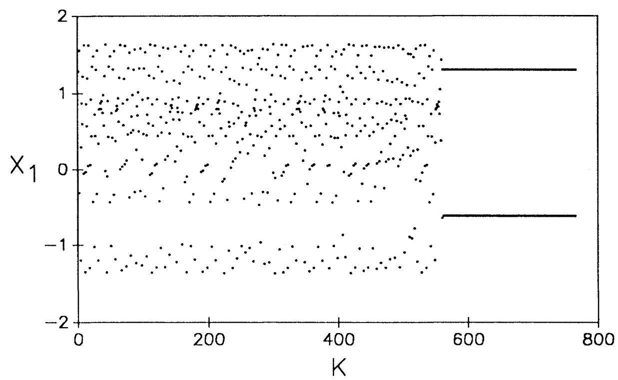

FIG. 3. Time series for the stabilization of a period-2 orbit. The initial condition is $(x_{10},x_{20})=(1,-1)$, and the perturbation bound $\delta^*=0.025$.

USE OF DELAY COORDINATES

For a large set of periodically forced systems, it is possible to derive an explicit discrete-time model (induced by a Poincaré section) which can be used to design nonlinear controllers [9]. In general, such models may present parametric and/or structural uncertainties. If one is able to estimate the size of such uncertainties, robust controllers can be derived [13] which account for model/reality deviation.

We now discuss the case where the dynamical equations are not known. In experimental studies of chaotic dynamical systems, delay coordinates are often used to represent the system state. It only requires measurement of the time series of a single scalar state variable. Let $x(t)$ be such a state variable and $Z(t)$ be the delay coordinate vector, which is given by

$$Z(t)=(x(t),x(t-T),x(t-2T),\ldots,x(t-MT)) ,$$

where T is the delay time. If n is the dimensionality of the dynamical system, for $M\geq 2n$ the vector $Z(t)$ is generically a global one-to-one representation of the system. In the presence of parameter variations u_i, delay coordinates lead to a map of the form [14]:

$$Z_{i+1}=G(Z_i,u_{i-r},\ldots,u_{i-1},u_i) . \tag{2}$$

Observe that the state of the system (2) depends on the last $r+1\geq 1$ values of the parameter u. Reyl *et al.* [15] used the model (2) to fit the return map of experimental NMR laser dynamics. This map was used to control the dynamics of a NMR laser by means of linear feedback.

Define the dummy coordinates $y_{k,i}=u_{i-k}$, $k=1,\ldots,r$. Then the map (2) can be written as the following augmented system:

$$\begin{aligned} Z_{i+1}&=G(Z_i,y_i,u_i) , \\ Y_{i+1}&=SY_i+Bu_i , \end{aligned} \tag{3}$$

where $Y_i=(y_{1,i},\ldots,y_{i,r})^*$, $B=(1,0,\ldots,0)^*$ ($*$ denotes transpose operation), and S is an $r\times r$ matrix, which is given by

$$S=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} .$$

In the coordinates (Z,Y), the map (3) depends only on the actual value of the parameter u_i. Given the fitted map (3), and if the system is controllable, it is possible to design a saturated nonlinear feedback $u_i=\mathcal{S}(F(Z_i,Y_i))$ which renders the dynamical system asymptotically stable to a periodic orbit.

In conclusion, we have shown that a saturated nonlinear feedback is able to stabilize chaotic systems. This methodology does not require that the objective trajectory $\{r_n\}$ be embedded in a chaotic attractor [2]. However, input saturation may introduce additional behaviors, such as multistability [12].

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